

A CATEGORIFICATION OF THE CHROMATIC SYMMETRIC FUNCTION

RADMILA SAZDANOVIĆ AND MARTHA YIP

ABSTRACT. The Stanley chromatic symmetric function X_G of a graph G is a symmetric function generalization of the chromatic polynomial, and has interesting combinatorial properties. We apply the ideas of Khovanov homology to construct a homology of graded \mathfrak{S}_n -modules, whose graded Frobenius series $\text{Frob}_G(q, t)$ reduces to the chromatic symmetric function at $q = t = 1$. This homology can be thought of as a categorification of the chromatic symmetric function, and provides a homological analogue of several familiar properties of X_G . In particular, the decomposition formula for X_G discovered recently by Orellana and Scott, and Guay-Paquet is lifted to a long exact sequence in homology.

1. INTRODUCTION

Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. A *proper colouring* of G is a function $\kappa : V(G) \rightarrow \mathbb{N}$ such that $\kappa(v_i) \neq \kappa(v_j)$ if an edge is incident to both v_i and v_j . The *chromatic polynomial* $\chi_G(x)$ of G is a polynomial such that for every $k \in \mathbb{N}$, $\chi_G(k)$ is the number of proper colourings of G with k colours.

Definition 1.1. The *chromatic symmetric function* of G is defined to be

$$(1) \quad X_G(\mathbf{x}) = X_G(x_1, x_2, \dots) = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)},$$

where the sum is over all proper colourings $\kappa : V(G) \rightarrow \mathbb{N}$ of G .

The polynomial X_G is a generalization of the chromatic polynomial χ_G in the sense that $X_G(1^k) = \chi_G(k)$ [Stanley 1995, Proposition 2.2], for $k \in \mathbb{N}$.

For each $i \in \mathbb{N}$, $\kappa^{-1}(i)$ is an independent set in G , so any permutation of \mathbb{N} that fixes all but finitely many elements gives another proper colouring of G , and therefore X_G is a symmetric function, homogeneous of degree n . Following the standard notation used by Macdonald, let p_λ and s_λ respectively denote the power sum symmetric functions and the Schur symmetric functions. Given a subset of edges $F \subseteq E(G)$, its *partition type* $\lambda(F)$ is the partition associated to the sizes of the connected components of the subgraph of G induced by the edge set F . The following formula [Stanley 1995,

Key words and phrases. symmetric functions, chromatic polynomial, Khovanov homology, Frobenius series, symmetric group representations, graph colouring.

Theorem 2.5] can be proved by an inclusion-exclusion argument.

$$(2) \quad X_G = \sum_{F \subseteq E(G)} (-1)^{|F|} p_{\lambda(F)}.$$

This formula forms the basis of our categorification process.

Categorification can be thought as a way of replacing an n -category by an $(n+1)$ -category; for example, lifting the Euler characteristic of a topological space to its homology. One of the most successful recent examples of categorification include link homology [Khovanov 2000], which is a new kind of link invariant that lifts the properties of the Jones polynomial and carries a rich algebraic structure. In this theory, every link is assigned bigraded homology groups whose Euler characteristic is the Jones polynomial, and additionally, link cobordisms are assigned homomorphisms of homology groups. This categorification has been successfully used in determining topological properties of knots, and gives a purely combinatorial proof [Rasmussen 2010] of the Milnor conjecture, also known as Kronheimer-Mrowka theorem.

Chromatic graph homology, developed by [Helme-Guizon and Rong 2005], is one of several categorifications of polynomial graph invariants. The construction follows that of Khovanov; a bigraded homology theory is associated to a graph and a commutative graded algebra in a way that its graded Euler characteristic is the chromatic polynomial evaluated at the q -dimension of the algebra. There are other categorifications of the chromatic polynomial, and all of them possess a long exact sequence of homology that lifts the deletion-contraction formula for the chromatic polynomial, in the same way that Khovanov homology lifts the skein relations for the Jones polynomial.

In this paper we apply a Khovanov-type construction to the chromatic symmetric function X_G . This process is described in Section 2. Every spanning subgraph of G is assigned a graded \mathfrak{S}_n -module, leading to a chain complex whose differential maps are defined based on the Boolean lattice structure of the set of spanning subgraphs of G . The Frobenius series $\text{Frob}_G(q, t)$ of the resulting bigraded homology, which we call the chromatic symmetric homology, specializes to the chromatic symmetric function, naturally expressed in the Schur basis.

In Section 3, we give the analogue of several familiar properties of the chromatic symmetric function on the categorified level. In particular, we consider the homology of graphs which contain a loop or multiple edges, and the homology of a disjoint union of graphs.

Unlike the chromatic polynomial χ_G , the chromatic symmetric function X_G does not satisfy a deletion-contraction formula, but as shown by [Orellana and Scott 2014] and [Guay-Paquet 2013], if G contains a 3-cycle, then X_G satisfies a recursive formula involving the deletion of two of the edges of the 3-cycle. In Section 4, we show how this recursive formula for X_G can be lifted to a long exact sequence in homology.

To conclude, we provide computations of homology for small graphs in Section 5. Finally, it is our hope that chromatic symmetric homology will be useful in addressing open problems regarding the chromatic symmetric function, such as the characterization of graphs whose X_G is Schur-positive ([Gasharov 1996]), and whether X_G is a complete invariant for trees ([Martin, Morin and Wagner 2008]). Also, we hope that this construction may shed light on the chromatic quasisymmetric function of [Shareshian and Wachs 2014].

1.1. Preliminaries. For more background on the theory of symmetric functions, we refer the reader to [Macdonald 1995], and for more background on the representation theory of the symmetric group, we refer the reader to [Fulton 1997].

A *partition* λ of n , denoted by $\lambda \vdash n$, is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ whose sum is n . Let \mathcal{P} denote the set of all partitions. For $i \in \mathbb{N}$, let $m_i(\lambda)$ be the multiplicity of i in λ , and let $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$.

For $k \in \mathbb{N}$, the *kth power sum symmetric function* is

$$p_k = p_k(x_1, x_2, \dots) = x_1^k + x_2^k + \dots,$$

and the *power sum symmetric function* indexed by λ is

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_r}.$$

Let $\Lambda_{\mathbb{C}}$ denote the space of symmetric functions in the indeterminates x_1, x_2, \dots . Then $\{p_\lambda \mid \lambda \in \mathcal{P}\}$ is a basis for $\Lambda_{\mathbb{C}}$.

Let \mathfrak{S}_n denote the symmetric group on n elements. The irreducible representations of \mathfrak{S}_n over \mathbb{C} are indexed by the partitions of n , and are called *Specht modules*. Let \mathcal{S}^λ denote the Specht module indexed by λ .

The *Schur symmetric function* indexed by λ satisfies

$$s_\lambda = \sum_{\mu} \chi^\lambda(\mu) z_\mu^{-1} p_\mu,$$

where $\chi^\lambda(\mu)$ is the character of the irreducible representation \mathcal{S}^λ , evaluated on a permutation of cycle type μ . The set $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ is another basis for $\Lambda_{\mathbb{C}}$. By duality, one obtains the Murnaghan-Nakayama rule [Macdonald 1995, I.7.8],

$$(3) \quad p_\mu = \sum_{\lambda} \chi^\lambda(\mu) s_\lambda,$$

and combining this with the formula in equation (2) yields a formula for X_G in terms of the Schur basis.

Remark 1.2. If $\{u_\lambda\}$ is a basis for the space of symmetric functions, then $X_G = \sum_{\lambda} a_\lambda u_\lambda$ is said to be *u-positive* if $a_\lambda \geq 0$ for all partitions λ . While it is clear that X_G is not *p-positive* in general, it turns out that X_G is *s-positive* for a certain class of graphs, although this is far from obvious following from the recipe for X_G provided above. [Gasharov 1996] obtained a remarkable combinatorial formula for X_G in the

Schur basis for the case when G is an incomparability graph of a $(\mathbf{3} + \mathbf{1})$ -free poset, and he used this to show that X_G is s -positive in this case.

The *Grothendieck group* R_n of representations of \mathfrak{S}_n is the free abelian group on the isomorphism classes $[\mathcal{S}^\lambda]$ of irreducible representations of \mathfrak{S}_n . Let $R = \bigoplus_{n \geq 0} R_n$. If $[V] \in R_a$ and $[W] \in R_b$, then multiplication in R is defined by

$$(4) \quad [V] \circ [W] = \left[\text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}} (V \otimes W) \right].$$

Given irreducible modules \mathcal{S}^λ and \mathcal{S}^μ , the Littlewood-Richardson Rule [Fulton 1997, §5.2 Corollary 3] is a combinatorial formula for computing their product

$$(5) \quad [\mathcal{S}^\lambda] \circ [\mathcal{S}^\mu] = \bigoplus_{\nu} c_{\lambda\mu}^{\nu} [\mathcal{S}^\nu],$$

where $c_{\lambda\mu}^{\nu}$ is the number of Littlewood-Richardson tableaux of shape ν/λ and weight μ . In other words, $c_{\lambda\mu}^{\nu}$ is the number of fillings of the skew shape ν/λ with $m_i(\mu)$ i 's, such that the filling is weakly increasing along rows, strictly increasing along columns, and the reading word obtained by reading the tableau from top to bottom and from right to left is a lattice word (ie. in every initial subword, i appears at least as often as $i + 1$ for every i .)

The morphism of graded rings given by sending the Specht modules to the Schur functions

$$(6) \quad \text{ch} : R \rightarrow \Lambda_{\mathbb{C}} : [\mathcal{S}^\lambda] \mapsto s_\lambda$$

is an isomorphism [Fulton 1997, §7.3 Theorem 1].

For $n \geq 1$, it is a consequence of the Murnaghan-Nakayama rule (3) that

$$(7) \quad \text{ch}^{-1}(p_n) = \sum_{i=0}^{n-1} (-1)^i \left[\mathcal{S}^{(n-i, 1^i)} \right],$$

is an alternating sum is over all hook-shaped partitions of n .

2. THE CONSTRUCTION

2.1. The state diagram. Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G) = \{e_1, \dots, e_m\}$.

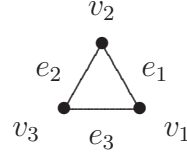
Definition 2.1. A *state* of G is a spanning subgraph with a subset of edges $F \subseteq E$. Let $|F|$ denote the number of edges in F .

A graph with m edges therefore has 2^m states. Let $Q(G)$ denote the set of all states of G . We identify a state F with the binary string $z_F = z_1 \cdots z_n$, where

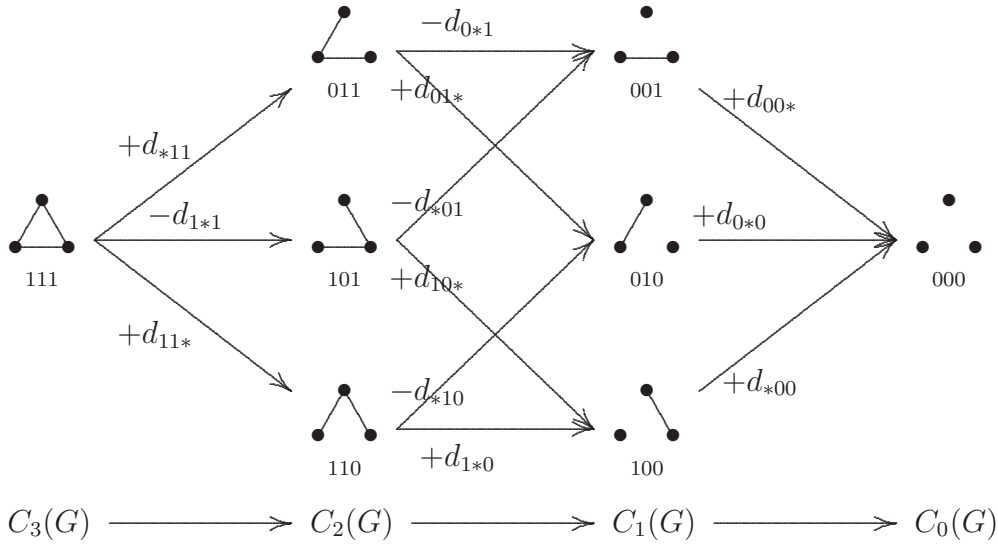
$$z_i = \begin{cases} 0, & \text{if } e_i \in F, \\ 1, & \text{if } e_i \notin F. \end{cases}$$

With this identification, the states of G form a Boolean lattice; that is, in the Hasse diagram of $Q(G)$, there is an edge from a state F to a state F' if and only if their associated binary strings differ in exactly one position. The Boolean lattice $Q(G)$ is graded by the number of edges in the states.

Example 2.2. Consider the 3-cycle $G = K_3$ with vertex set $V = \{v_1, v_2, v_3\}$ and edge set $E = \{e_1, e_2, e_3\}$.



The eight states of K_3 and their associated binary strings are shown below, together with the signed per-edge maps (to be defined in Section 2.3):



2.2. The graded modules. We shall assign a graded \mathfrak{S}_n -module to each state. Let \mathcal{S}^λ denote the irreducible \mathfrak{S}_n -module indexed by the partition $\lambda \vdash n$. For $a = 1$, let $\mathcal{L}_a = \mathcal{S}^{(a)}$, and for $a \in \mathbb{Z}_{\geq 2}$, let \mathcal{L}_a denote the graded \mathfrak{S}_n -module

$$(8) \quad \mathcal{L}_a = \bigoplus_{i=0}^{a-1} \mathcal{S}^{(a-i, 1^i)}.$$

Since $\bigwedge^i \mathcal{S}^{(a-1, 1)} \cong \mathcal{S}^{(a-i, 1^i)}$, we may also think of \mathcal{L}_a as the exterior algebra $\bigwedge^* \mathcal{S}^{(a-1, 1)}$.

Suppose $F \subseteq E(G)$ is a state with r connected components B_1, \dots, B_r of sizes b_1, \dots, b_r respectively. To F , we assign the graded \mathfrak{S}_n -module

$$(9) \quad \mathcal{M}_F = \text{Ind}_{\mathfrak{S}_{B_1} \times \dots \times \mathfrak{S}_{B_r}}^{\mathfrak{S}_{V(G)}} (\mathcal{L}_{b_1} \otimes \dots \otimes \mathcal{L}_{b_r}),$$

where $\mathfrak{S}_{B_1} \times \dots \times \mathfrak{S}_{B_r}$ is a Young subgroup of $\mathfrak{S}_{V(G)} \cong \mathfrak{S}_n$.

For the sake of simpler notation, we will sometimes use the shorthand

$$(10) \quad \text{Ind}_{B_1|\dots|B_r} = \text{Ind}_{\mathfrak{S}_{B_1} \times \dots \times \mathfrak{S}_{B_r}}^{\mathfrak{S}_{V(G)}}.$$

Definition 2.3. For $i \geq 0$, the i th chain module for G is

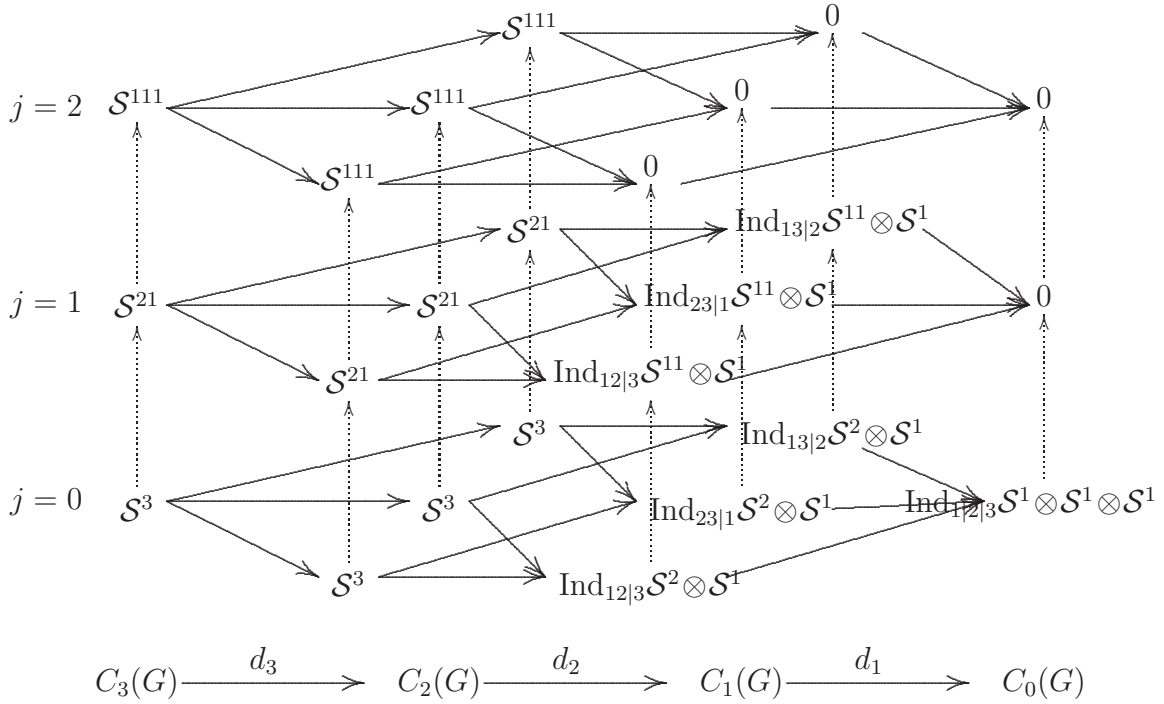
$$C_i(G) = \bigoplus_{|F|=i} \mathcal{M}_F.$$

More precisely, since $\mathcal{M}_F = \bigoplus_{j \geq 0} (\mathcal{M}_F)_j$ is graded, then for $i, j \geq 0$, we can define

$$C_{i,j}(G) = \bigoplus_{|F|=i} (\mathcal{M}_F)_j.$$

Observe that $C_{i,j}(G) = 0$ for all $j > i$.

Example 2.4. Continuing the K_3 example, the graded \mathfrak{S}_3 -modules corresponding to the states are:



2.3. The per-edge maps. So far, the modules \mathcal{M}_F are defined up to isomorphism. In the following, we show that for each edge ε in the Hasse diagram of $Q(G)$ from a state F to a state F' , there is an obvious choice of a graded \mathfrak{S}_n -module morphism $d_\varepsilon : \mathcal{M}_F \rightarrow \mathcal{M}_{F'}$. These are the *per-edge maps*.

There are two cases to consider. Suppose $F' = F - e$ where $e \in E(G)$.

Case 1. The edge e is incident to vertices in the same connected component of F' .

Since \mathcal{M}_F and $\mathcal{M}_{F'}$ are canonically isomorphic, we define $d_e : \mathcal{M}_F \rightarrow \mathcal{M}_{F'}$ to be the identity map.

Case 2. The edge e is incident to vertices in different connected components of F' .

First, consider the simplest case where F consists of one connected component, and F' consists of two connected components A and B . Suppose $|A| = a$ and $|B| = b$, so that $a + b = n$. Given that $\mathcal{M}_F \cong \mathcal{L}_n$ and $\mathcal{M}_{F'} \cong \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}(\mathcal{L}_a \otimes \mathcal{L}_b)$, we shall show that there is a canonical choice of an element in $\text{Hom}_{\mathfrak{S}_n}(\mathcal{M}_F, \mathcal{M}_{F'})$.

Lemma 2.5. *Let $\mathbf{1}_W$ denote the trivial representation of the group W . Then*

$$\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{S}^{(n-1,1)} \cong ((\mathcal{S}^{(a-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_b}) \oplus (\mathbf{1}_{\mathfrak{S}_a} \otimes \mathcal{S}^{(b-1,1)})) \oplus \mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b}.$$

Proof. The permutation representation of \mathfrak{S}_n is the module

$$\mathcal{M}^{(n-1,1)} \cong \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathbf{1}_{\mathfrak{S}_{n-1}}.$$

Since $\mathcal{M}^{(n-1,1)} \cong \mathcal{S}^{(n-1,1)} \oplus \mathcal{S}^{(n)}$ and $\mathcal{S}^{(n)} \cong \mathbf{1}_{\mathfrak{S}_n}$, then

$$\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{M}^{(n-1,1)} \cong (\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{S}^{(n-1,1)}) \oplus \mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b}.$$

On the other hand,

$$\begin{aligned} \text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{M}^{(n-1,1)} &\cong (\mathcal{M}^{(a-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_b}) \oplus (\mathbf{1}_{\mathfrak{S}_a} \otimes \mathcal{M}^{(b-1,1)}) \\ &\cong (\mathcal{S}^{(a-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_b}) \oplus (\mathbf{1}_{\mathfrak{S}_a} \otimes \mathcal{S}^{(b-1,1)}) \oplus (\mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b})^{\oplus 2}. \end{aligned}$$

Comparing the two expressions for $\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{M}^{(n-1,1)}$ yields the result. \square

Lemma 2.6. *Let $\mathcal{T} = (\mathcal{S}^{(a-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_b}) \oplus (\mathbf{1}_{\mathfrak{S}_a} \otimes \mathcal{S}^{(b-1,1)})$. Then*

$$\text{Hom}_{\mathfrak{S}_n}(\mathcal{M}_F, \mathcal{M}_{F'}) \cong \text{Hom}_{\mathfrak{S}_a \times \mathfrak{S}_b}(\wedge^* \mathcal{T} \oplus (\wedge^* \mathcal{T})[1], \wedge^* \mathcal{T}),$$

where $(\wedge^* \mathcal{T})[1]$ denotes the first translate of $\wedge^* \mathcal{T}$.

Proof. By Frobenius Reciprocity,

$$\begin{aligned} \text{Hom}_{\mathfrak{S}_n}(\mathcal{M}_F, \mathcal{M}_{F'}) &= \text{Hom}_{\mathfrak{S}_n}(\mathcal{L}_n, \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n}(\mathcal{L}_a \otimes \mathcal{L}_b)) \\ &= \text{Hom}_{\mathfrak{S}_a \times \mathfrak{S}_b}(\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n}(\mathcal{L}_n), \mathcal{L}_a \otimes \mathcal{L}_b) \end{aligned}$$

is a natural isomorphism of vector spaces. Note that

$$\mathcal{L}_a \otimes \mathcal{L}_b = \wedge^* \mathcal{S}^{(a-1,1)} \otimes \wedge^* \mathcal{S}^{(b-1,1)} = \wedge^* (\mathcal{S}^{(a-1,1)} \oplus \mathcal{S}^{(b-1,1)}) \cong \wedge^* \mathcal{T}.$$

Also, by Lemma 2.5,

$$\begin{aligned} \text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{L}_n &= \text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} (\wedge^* \mathcal{S}^{(n-1,1)}) = \wedge^* (\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n} \mathcal{S}^{(n-1,1)}) \\ &= \wedge^* (\mathcal{T} \oplus \mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b}) = \wedge^* \mathcal{T} \otimes \wedge^* \mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b} \\ &= \wedge^* \mathcal{T} \otimes (\mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b} \oplus \mathbf{1}_{\mathfrak{S}_a \times \mathfrak{S}_b}[1]) \\ &= \wedge^* \mathcal{T} \oplus (\wedge^* \mathcal{T})[1]. \end{aligned}$$

From this, the result follows. \square

Since $\text{Hom}_{\mathfrak{S}_n}(\mathcal{M}_F, \mathcal{M}_{F'}) = \text{Hom}_{\mathfrak{S}_a \times \mathfrak{S}_b}(\wedge^* \mathcal{T} \oplus (\wedge^* \mathcal{T})[1], \wedge^* \mathcal{T})$, we choose the element $d_\varepsilon \in \text{Hom}_{\mathfrak{S}_n}(\mathcal{M}_F, \mathcal{M}_{F'})$ to be the map that corresponds to the $(\mathfrak{S}_a \times \mathfrak{S}_b)$ -module map that is identity on $\wedge^* \mathcal{T}$, and zero on $(\wedge^* \mathcal{T})[1]$.

It remains to consider the general case when F has more than one connected component. The definition of the per-edge map in this case is achieved through induction on the two-component case.

Suppose F is a state with r connected components B_1, \dots, B_r of sizes b_1, \dots, b_r . Further suppose that the removal of the edge $e \in E(G)$ decomposes B_r into two components A and B of sizes a and b respectively.

Let $d_\zeta : \mathcal{L}_{a+b} \rightarrow \text{Ind}_{\mathfrak{S}_A \times \mathfrak{S}_B}^{\mathfrak{S}_{B_r}}(\mathcal{L}_a \otimes \mathcal{L}_b)$ be the per-edge map defined previously, and let $\mathcal{N} = \mathcal{L}_{b_1} \otimes \dots \otimes \mathcal{L}_{b_{r-1}}$. The map $d_\varepsilon : \mathcal{M}_F \rightarrow \mathcal{M}_{F'}$ is chosen to be

$$(11) \quad d_\varepsilon = \text{Ind}_{\mathfrak{S}_{B_1} \times \dots \times \mathfrak{S}_{B_{r-1}} \times \mathfrak{S}_{B_r}}^{\mathfrak{S}_{V(G)}}(\text{id}_{\mathcal{N}} \otimes d_\zeta).$$

2.3.1. Signed per-edge maps. Lastly, we define signed per-edge maps. Suppose the associated binary strings z_F and $z_{F'}$ differ in the j th position. Identify ε with the sequence $\varepsilon_1 \dots \varepsilon_m$ defined by $\varepsilon_j = *$, and $\varepsilon_i = (z_F)_i = (z_{F'})_i$ for $i \neq j$. See Example 2.2.

Definition 2.7. If an edge $\varepsilon = \varepsilon_1 \dots \varepsilon_m$ in the Hasse diagram of $Q(G)$ has k ones after the $*$ entry, then the *signed per-edge map* that corresponds to ε is

$$\text{sgn}(\varepsilon)d_\varepsilon = (-1)^k d_\varepsilon.$$

Remark 2.8. This choice of a sign convention on the edges of $Q(G)$ guarantees that any induced sublattice isomorphic to a square will have three edges of the same sign and one edge of the opposite sign, which is crucial in the proof that d is indeed a differential map, as will be seen in the next section.

2.4. The chain complex of graded modules. Now that we have assigned a graded \mathfrak{S}_n -module \mathcal{M}_F to each state F of G , and defined signed per-edge maps, we can make the following definition.

Definition 2.9. For $i \geq 0$, define $d_i : C_i(G) \rightarrow C_{i-1}(G)$ by letting

$$d_i = \sum_{\varepsilon} \text{sgn}(\varepsilon)d_\varepsilon,$$

where the sum is over all edges ε in the Hasse diagram of $Q(G)$ joining a state with i edges to a state with $i - 1$ edges. We also define $d_{i,j} : C_{i,j}(G) \rightarrow C_{i-1,j}(G)$ to be the map d_i in the j th grading.

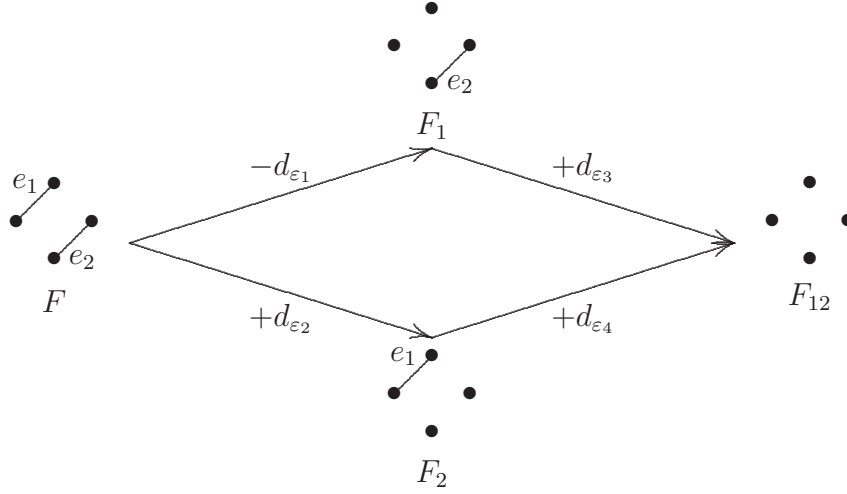
Proposition 2.10. *This defines a differential; that is, $d^2 = 0$.*

Proof. Since d is defined via per-edge maps, it suffices to show that $d^2 = 0$ on the per-edge maps. This is achieved through case checking; up to signs, the three cases

that must be considered depend on how the removal of two edges from a state of G will disconnect graph components.

Let F be a state of G , and suppose e_1 and e_2 are the two edges to be removed from F . We use the notation $F_i = F - e_i$, and $F_{12} = F - e_1 - e_2$. Label the four edges in the Hasse diagram of $Q(G)$ as in the following diagram. It is straightforward to verify that the sign convention on the per-edge maps forces each ‘square’ in the Hasse diagram $Q(G)$ to have three of one sign, and one of the opposite sign.

We focus on the case with the signs indicated in the following diagram; the other three cases are completely analogous.



Now we consider the three cases which depend on how the removal of e_1 and e_2 disconnects graph components. Let $\kappa(F)$ denote the number of connected components of F . Observe that $\kappa(F_{12}) \leq \kappa(F) + 2$. In each case, we verify that $d_{\epsilon_4} \circ d_{\epsilon_2} - d_{\epsilon_3} \circ d_{\epsilon_1} = 0$.

Case 1. $\kappa(F_{12}) = \kappa(F)$. In this case, the modules corresponding to all four states are equal, and each per-edge map is the identity map (up to signs), so for any $x \in \mathcal{M}_F$,

$$x \mapsto (-x, x) \mapsto x - x = 0.$$

Case 2. $\kappa(F_{12}) = \kappa(F) + 1$. There are two subcases to consider. First, suppose removing e_1 disconnects a graph component, but removing e_2 does not. Since the removal of e_2 does not disconnect components, then $\mathcal{M}_F = \mathcal{M}_{F_2}$, $\mathcal{M}_{F_1} = \mathcal{M}_{F_{12}}$, and the two per-edge maps d_{ϵ_2} and d_{ϵ_3} are the identity maps. If the removal of e_1 disconnects a component of size n into components of sizes a and b , then the maps d_{ϵ_1} and d_{ϵ_4} are each induced from the same map $\mathcal{L}_n \rightarrow \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n}(\mathcal{L}_a \otimes \mathcal{L}_b)$. Therefore, $d_{\epsilon_1} = d_{\epsilon_4}$, and

$$d_{\epsilon_4} \circ \text{id}_{\mathcal{M}_F} - \text{id}_{\mathcal{M}_{F_1}} \circ d_{\epsilon_1} = 0.$$

Second, suppose that removing each of e_1 or e_2 does not disconnect a graph component, but that removing both edges simultaneously disconnects a component. Then

$\mathcal{M}_F = \mathcal{M}_{F_1} = \mathcal{M}_{F_2}$, and the two per-edge maps d_{ε_1} and d_{ε_2} are the identity maps up to sign. If the removal of e_1 and e_2 disconnects a component of size n into components of sizes a and b , then the maps d_{ε_3} and d_{ε_4} are each induced from the same map $\mathcal{L}_n \rightarrow \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_n}(\mathcal{L}_a \otimes \mathcal{L}_b)$. Therefore, $d_{\varepsilon_3} = d_{\varepsilon_4}$, and

$$d_{\varepsilon_4} \circ \text{id}_{\mathcal{M}_F} - d_{\varepsilon_3} \circ \text{id}_{\mathcal{M}_F} = 0.$$

Case 3. $\kappa(F_{12}) = \kappa(F) + 2$. There are two subcases to consider. First, suppose e_1 and e_2 are in different components A and B of F , respectively. Let $|A| = a$ and $|B| = b$. For simplicity, we assume that F only has two components; the general case is handled by induction (11). So $\mathcal{M}_A = \mathcal{L}_a$, $\mathcal{M}_B = \mathcal{L}_b$, and $\mathcal{M}_F = \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}}(\mathcal{L}_a \otimes \mathcal{L}_b)$. Let $f : \mathcal{M}_A \rightarrow \mathcal{M}_{A-e_1}$ and $g : \mathcal{M}_B \rightarrow \mathcal{M}_{B-e_2}$. We then have the following maps of $(\mathfrak{S}_a \times \mathfrak{S}_b)$ -modules:

$$\begin{array}{ccc} & \mathcal{M}_{A-e_1} \otimes \mathcal{M}_B & \\ \begin{array}{c} \nearrow -f \otimes \text{id} \\ \searrow +\text{id} \otimes g \end{array} & & \begin{array}{c} \nwarrow +\text{id} \otimes g \\ \nearrow +f \otimes \text{id} \end{array} \\ \mathcal{M}_A \otimes \mathcal{M}_B & & \mathcal{M}_{A-e_1} \otimes \mathcal{M}_{B-e_2} \\ & \mathcal{M}_A \otimes \mathcal{M}_{B-e_2} & \end{array}$$

so that

$$(f \circ \text{id}_{\mathcal{M}_A}) \otimes (\text{id}_{\mathcal{M}_{B-e_2}} \circ g) - (\text{id}_{\mathcal{M}_{A-e_1}} \circ f) \otimes (g \circ \text{id}_{\mathcal{M}_B}) = 0.$$

Induce this to \mathfrak{S}_{a+b} -modules to conclude that $d_{\varepsilon_4} \circ d_{\varepsilon_2} - d_{\varepsilon_3} \circ d_{\varepsilon_1} = 0$.

Lastly, suppose e_1 and e_2 are in the same component of F . Again for simplicity, we assume that F consists of only one component, and the general case is handled by induction. Suppose F_{12} consists of the three components A_1, A_2 , and A_3 of sizes a, b and c respectively, while F_1 consists of the components A_1 and $A_2 \cup A_3$ and F_2 consists of the components $A_1 \cup A_2$ and A_3 . Let $n = a + b + c$. We have the following maps of \mathfrak{S}_n -modules:

$$\begin{array}{ccc} & \mathcal{M}_{F_1} = \text{Ind}(\mathcal{L}_a \otimes \mathcal{L}_{b+c}) & \\ \begin{array}{c} \nearrow -d_{\varepsilon_1} \\ \searrow +d_{\varepsilon_2} \end{array} & & \begin{array}{c} \nwarrow +d_{\varepsilon_3} \\ \nearrow +d_{\varepsilon_4} \end{array} \\ \mathcal{M}_F = \mathcal{L}_n & & \mathcal{M}_{F_{12}} = \text{Ind}(\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c) \\ & \mathcal{M}_{F_2} = \text{Ind}(\mathcal{L}_{a+b} \otimes \mathcal{L}_c) & \end{array}$$

Recall from equation 11 that the map d_{ε_3} is induced from $\text{id}_{\mathcal{L}_a} \otimes (\mathcal{L}_{b+c} \rightarrow \text{Ind}(\mathcal{L}_b \otimes \mathcal{L}_c))$, so by Frobenius reciprocity, the \mathfrak{S}_n -module map $d_{\varepsilon_3} \circ d_{\varepsilon_1}$ corresponds to the $(\mathfrak{S}_a \times \mathfrak{S}_{b+c})$ -module map

$$\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_{b+c}}^{\mathfrak{S}_n}(\mathcal{L}_n) \longrightarrow \mathcal{L}_a \otimes \mathcal{L}_{b+c} \longrightarrow \mathcal{L}_a \otimes \text{Ind}_{\mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_{b+c}}(\mathcal{L}_b \otimes \mathcal{L}_c).$$

Applying Frobenius reciprocity one more time, this map then in turn corresponds to the $(\mathcal{L}_a \times \mathcal{L}_b \times \mathcal{L}_c)$ -module map

$$\psi : \text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_n}(\mathcal{L}_n) \longrightarrow \mathcal{L}_a \otimes \text{Res}_{\mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_{b+c}}(\mathcal{L}_{b+c}) \longrightarrow \mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c.$$

Letting $\mathcal{L}_b \otimes \mathcal{L}_c = \wedge^* \mathcal{U}$, where $\mathcal{U} = (\mathcal{S}^{(b-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_c}) \oplus (\mathbf{1}_{\mathfrak{S}_b} \otimes \mathcal{S}^{(c-1,1)})$, then

$$\mathcal{L}_a \otimes \text{Res}_{\mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_{b+c}}(\mathcal{L}_{b+c}) = \mathcal{L}_a \otimes (\wedge^* \mathcal{U} \oplus \wedge^* \mathcal{U}[1]).$$

Also, letting $\mathcal{L}_a \otimes \mathcal{L}_{b+c} = \wedge^* \mathcal{T}$, where $\mathcal{T} = (\mathcal{S}^{(a-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_{b+c}}) \oplus (\mathbf{1}_{\mathfrak{S}_a} \otimes \mathcal{S}^{(b+c-1,1)})$, then

$$\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_n}(\mathcal{L}_n) = (\mathcal{L}_a \otimes \wedge^* \mathcal{U}) \oplus (\mathcal{L}_a \otimes \wedge^* \mathcal{U})^{\oplus 2} [1] \oplus (\mathcal{L}_a \otimes \wedge^* \mathcal{U})[2],$$

therefore, the composite map $\psi : \text{Res}(\mathcal{L}_n) \rightarrow \mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c$ is the identity map on $\mathcal{L}_a \otimes \wedge^* \mathcal{U}$, and is the zero map on the remaining factors.

Similarly, the \mathfrak{S}_n -module map $d_{\varepsilon_4} \circ d_{\varepsilon_2}$ corresponds to the $(\mathcal{L}_a \times \mathcal{L}_b \times \mathcal{L}_c)$ -module map

$$\theta : \text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_n}(\mathcal{L}_n) \longrightarrow \text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}}(\mathcal{L}_{a+b}) \otimes \mathcal{L}_c \longrightarrow \mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c,$$

and letting $\mathcal{L}_a \otimes \mathcal{L}_b = \wedge^* \mathcal{W}$ where $\mathcal{W} = (\mathcal{S}^{(a-1,1)} \otimes \mathbf{1}_{\mathfrak{S}_b}) \oplus (\mathbf{1}_{\mathfrak{S}_a} \otimes \mathcal{S}^{(b-1,1)})$, then

$$\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_b \times \mathfrak{S}_c}^{\mathfrak{S}_n}(\mathcal{L}_n) = (\wedge^* \mathcal{W} \otimes \mathcal{L}_c) \oplus (\wedge^* \mathcal{W} \otimes \mathcal{L}_c)^{\oplus 2} [1] \oplus (\wedge^* \mathcal{W} \otimes \mathcal{L}_c)[2],$$

and the composite map $\theta : \text{Res}(\mathcal{L}_n) \rightarrow \mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c$ is the identity map on $\wedge^* \mathcal{W} \otimes \mathcal{L}_c$, and is the zero map on the remaining factors.

Finally, since $\wedge^* \mathcal{W} \otimes \mathcal{L}_c = \mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c = \mathcal{L}_a \otimes \wedge^* \mathcal{U}$, we conclude that $\psi = \theta$, and by Frobenius reciprocity, it follows that $d_{\varepsilon_3} \circ d_{\varepsilon_1} = d_{\varepsilon_4} \circ d_{\varepsilon_2}$. \square

2.5. Decategorification via the Frobenius series. In this section, we show that the chromatic symmetric homology categorifies the chromatic symmetric function. The decategorification process amounts to computing the bigraded Frobenius series of the homology at $q = t = 1$.

Definition 2.11. For $i, j \geq 0$, the (i, j) th homology of G is

$$H_{i,j}(G) = \ker d_{i,j} / \text{im } d_{i+1,j}, \quad \text{so that} \quad H_i(G) = \bigoplus_{j \geq 0} H_{i,j}(G).$$

The bigraded *Frobenius series* of $H_*(G) = \bigoplus_{i,j \geq 0} H_{i,j}(G)$ is

$$\text{Frob}_G(q, t) = \sum_{i,j \geq 0} (-1)^{i+j} t^i q^j \text{ch}(H_{i,j}(G)),$$

where $\text{ch} : R \rightarrow \Lambda_{\mathbb{C}} : [\mathcal{S}^\lambda] \mapsto s_\lambda$ is the Frobenius characteristic map from the Grothendieck ring of representations to the ring of symmetric functions defined in equation (6).

Lemma 2.12. *For any graph G ,*

$$\sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G)) = \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(C_{i,j}(G)).$$

Proof. This proof is similar to that of the Euler characteristic of chain complexes. In the following, we supply the proof along with the necessary minor adaptations.

Any short exact sequence of \mathfrak{S}_n -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is split exact, so $B \cong A \oplus C$ and $\text{ch}(B) = \text{ch}(A) + \text{ch}(C)$.

Consider the chain complex

$$0 \longrightarrow C_{m,j}(G) \xrightarrow{d_{m,j}} C_{m-1,j}(G) \longrightarrow \cdots \longrightarrow C_{1,j}(G) \xrightarrow{d_{1,j}} C_{0,j}(G) \longrightarrow 0$$

for each $j \geq 0$. Let $Z_{i,j}(G) = \ker d_{i,j}$ and $B_{i,j}(G) = \text{im } d_{i+1,j}$. For $i, j \geq 0$, we have short exact sequences $0 \rightarrow Z_{i,j} \rightarrow C_{i,j} \rightarrow B_{i-1,j} \rightarrow 0$ and $0 \rightarrow B_{i,j} \rightarrow Z_{i,j} \rightarrow H_{i,j} \rightarrow 0$, where $B_{-1,j}$ is understood to be zero. Thus

$$\text{ch}(C_{i,j}) = \text{ch}(Z_{i,j}) + \text{ch}(B_{i-1,j}) = \text{ch}(H_{i,j}) + \text{ch}(B_{i,j}) + \text{ch}(B_{i-1,j}).$$

Multiplying this by $(-1)^{i+j}$ and summing over all $i, j \geq 0$ gives the desired result. \square

Theorem 2.13. *Chromatic symmetric homology categorifies the chromatic symmetric function. That is, for any graph G ,*

$$\text{Frob}_G(1, 1) = X_G.$$

Proof. Following Lemma 2.12, we compute

$$\begin{aligned} \text{Frob}_G(1, 1) &= \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G)) \\ &= \sum_{i \geq 0} (-1)^i \left(\sum_{j \geq 0} (-1)^j \text{ch}(C_{i,j}(G)) \right) && \text{by Lemma 2.12} \\ &= \sum_{i \geq 0} (-1)^i \sum_{\substack{F \subseteq E(G): \\ |F|=i}} p_{\lambda(F)} \\ &= X_G, && \text{by Equation (2).} \end{aligned}$$

\square

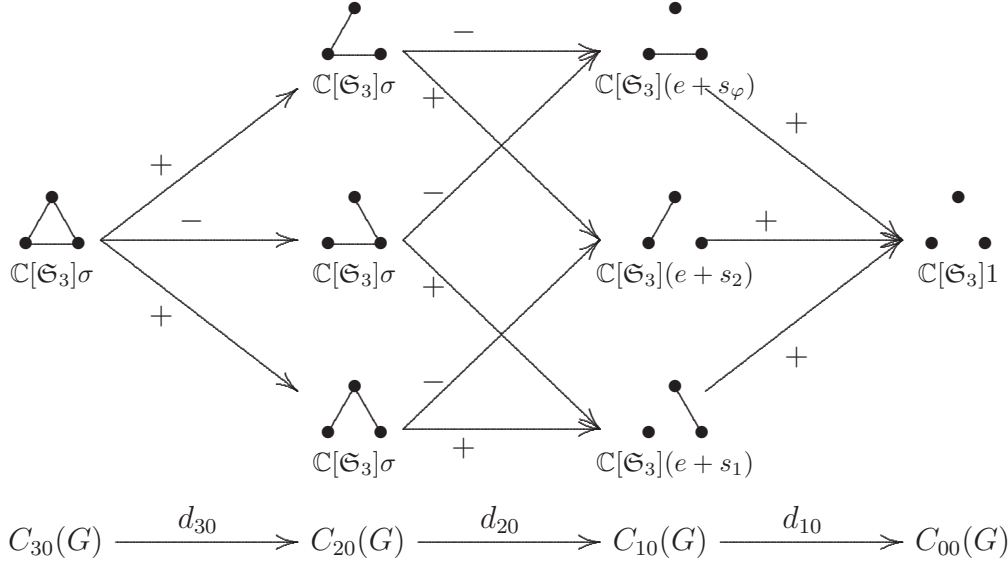
Example 2.14. We finish the K_3 example by computing its homology and its Frobenius series. Continuing from Example 2.4, recall that $\mathfrak{S}_3 = \langle s_1, s_2 \mid s_1^2 = e, s_2^2 = e, (s_1 s_2)^3 = e \rangle$. For convenience, we let $s_\varphi = s_1 s_2 s_1 = s_2 s_1 s_2$. Then $\text{Ind}_{1|2|3}(\mathcal{S}^1 \otimes \mathcal{S}^1 \otimes \mathcal{S}^1)$ is the regular representation of \mathfrak{S}_3 , so we choose the (ordered) basis

$$\begin{aligned} \text{Ind}_{1|2|3}(\mathcal{S}^1 \otimes \mathcal{S}^1 \otimes \mathcal{S}^1) &\cong \mathbb{C}[\mathfrak{S}_3] \\ &\cong \mathcal{S}^3 \oplus (\mathcal{S}^{21})^{\oplus 2} \oplus \mathcal{S}^{111} \\ &= \text{span}_{\mathbb{C}}\{e, s_1, s_2, s_2 s_1, s_1 s_2, s_1 s_2 s_1\}. \end{aligned}$$

We also choose these ordered bases for the following modules:

$$\begin{aligned} \text{Ind}_{23|1}(\mathcal{S}^2 \otimes \mathcal{S}^1) &= \mathbb{C}[\mathfrak{S}_3](e + s_\varphi)s_1 = \text{span}_{\mathbb{C}}\{e + s_2, s_1 + s_1s_2, s_2s_1 + s_\varphi\}, \\ \text{Ind}_{13|2}(\mathcal{S}^2 \otimes \mathcal{S}^1) &= \mathbb{C}[\mathfrak{S}_3](e + s_\varphi) = \text{span}_{\mathbb{C}}\{e + s_\varphi, s_1 + s_2s_1, s_2 + s_1s_2\}, \\ \text{Ind}_{12|3}(\mathcal{S}^2 \otimes \mathcal{S}^1) &= \mathbb{C}[\mathfrak{S}_3](e + s_\varphi)s_2 = \text{span}_{\mathbb{C}}\{e + s_1, s_2 + s_2s_1, s_1s_2 + s_\varphi\}. \end{aligned}$$

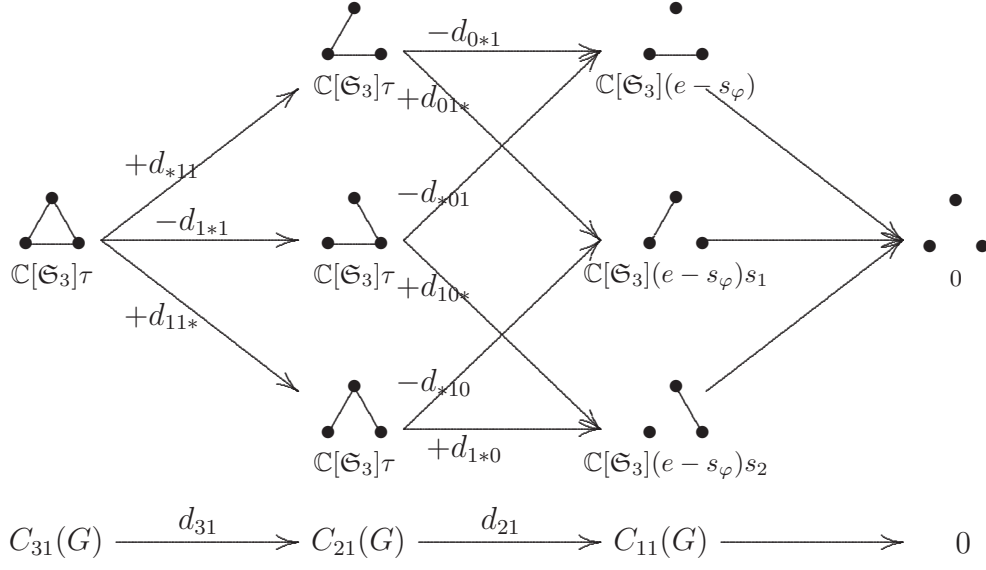
Also, choose $\mathcal{S}^3 = \mathbb{C}[\mathfrak{S}_3]\sigma = \text{span}_{\mathbb{C}}\{\sigma\}$, where $\sigma = e + s_1 + s_2 + s_2s_1 + s_1s_2 + s_\varphi$ is a Young symmetrizer. Then every per-edge map in the zeroth grading is an inclusion map:



Thus $\ker d_{30} \cong 0$, $\text{im } d_{30} = \ker d_{20} \cong \mathcal{S}^3$, $\text{im } d_{20} \cong (\mathcal{S}^3)^{\oplus 2}$, $\ker d_{10} \cong (\mathcal{S}^3)^{\oplus 2} \oplus \mathcal{S}^{21}$, $\text{im } d_{10} \cong \mathcal{S}^3 \oplus (\mathcal{S}^{21})^{\oplus 2}$, and $\ker d_{00} \cong \mathbb{C}[\mathfrak{S}_3]$.

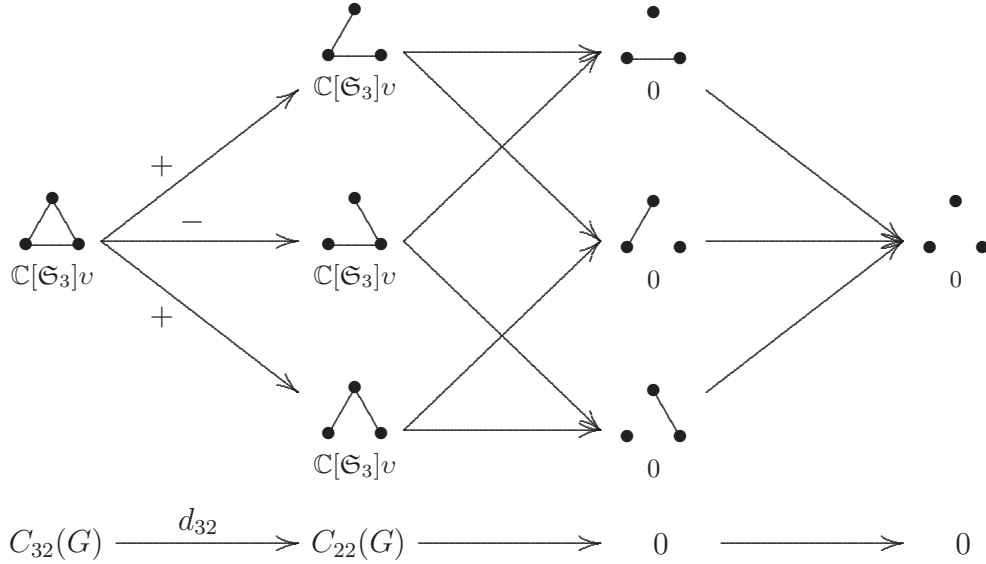
In the first grading, we choose the ordered basis $\{e - s_\varphi, s_1 - s_2s_1, s_2 - s_1s_2\}$ for $\text{Ind}_{13|2}(\mathcal{S}^{11} \otimes \mathcal{S}^1) = \mathbb{C}[\mathfrak{S}_3](e - s_\varphi)$. It follows that $\text{Ind}_{23|1}(\mathcal{S}^{11} \otimes \mathcal{S}^1) = \mathbb{C}[\mathfrak{S}_3](e - s_\varphi)s_1$ and $\text{Ind}_{12|3}(\mathcal{S}^{11} \otimes \mathcal{S}^1) = \mathbb{C}[\mathfrak{S}_3](e - s_\varphi)s_2$. Let $\tau = (s_2 + s_2s_1)(e - s_\varphi)$ be a Young symmetrizer in $\text{Ind}_{13|2}(\mathcal{S}^{11} \otimes \mathcal{S}^2)$, and choose $\mathcal{S}^{21} = \mathbb{C}[\mathfrak{S}_3]\tau$. Then the per-edge maps $d_{*11}, d_{1*1}, d_{11*}, d_{0*1}, d_{*01}$ are inclusion maps (up to sign). Also, $d_{01*}(\tau) = d_{*10}(\tau) = \tau s_1$,

while $d_{10*}(\tau) = d_{1*0}(\tau) = \tau s_2$.



Thus $\ker d_{31} \cong 0$, $\operatorname{im} d_{31} = \ker d_{21} \cong \mathcal{S}^{21}$, $\operatorname{im} d_{21} = (\mathcal{S}^{21})^{\oplus 2}$, and $\ker d_{11} \cong (\mathcal{S}^{21} \oplus \mathcal{S}^{111})^{\oplus 3}$.

Lastly, in the second grading, let $v = e - s_1 - s_2 + s_2 s_1 + s_1 s_2 - s_\varphi$, so that $\mathcal{S}^{111} = \mathbb{C}[\mathfrak{S}_3]v = \operatorname{span}_{\mathbb{C}}\{v\}$. Then



So $\ker d_{32} \cong 0$, $\operatorname{im} d_{32} \cong \mathcal{S}^{111}$, and $\ker d_{32} = (\mathcal{S}^{111})^{\oplus 2}$.

In summary, the bigraded homology of $G = K_3$ is

$$\begin{array}{c|ccc} 2 & (\mathcal{S}^{111})^{\oplus 2} & & \\ 1 & 0 & \mathcal{S}^{21} \oplus (\mathcal{S}^{111})^{\oplus 3} & \\ 0 & 0 & \mathcal{S}^{21} & \mathcal{S}^{111} \\ \hline & H_2(K_3) & H_1(K_3) & H_0(K_3) \end{array}$$

and the bigraded Frobenius series is

$$\begin{aligned} \text{Frob}_{K_3}(q, t) &= 2q^2t^2s_{111} + qt(s_{21} + 3s_{111}) - ts_{21} + s_{111} \\ &= (1 - q)ts_{21} + (1 + qt)(1 + 2qt)s_{111} \\ &= (1 - q)tm_{21} + (1 - 2t + 5qt + 2q^2t^2)m_{111}. \end{aligned}$$

The specializations are $\text{Frob}_{K_3}(1, t) = (1 + t)(1 + 2t)s_{111}$, and $\text{Frob}_{K_3}(1, 1) = 6s_{111}$.

3. PROPERTIES OF $H_*(G)$

3.1. Loops and multiple edges.

Proposition 3.1. *If G contains a loop, then $H_*(G) = 0$.*

Proof. Suppose G has m edges $\{e_1, \dots, e_m\}$, and $e_m = \ell$ is a loop in G . Let (U_*, d_*^U) be the chain complex obtained by restricting $(C_*(G), d_*)$ to the modules indexed by states which contain ℓ , and let $A_* = U_*[1]$ be the first translate of U_* so that $A_i = U_{i+1}$ for all $i \geq 0$. Let (B_*, d_*^B) be the chain complex obtained by restricting $(C_*(G), d_*)$ to the modules indexed by states which do not contain ℓ .

Observe that the only per-edge maps from modules in A_* to modules in B_* are of the form $\mathcal{M}_F \rightarrow \mathcal{M}_{F-\ell}$, and each of these maps is the identity map as $\mathcal{M}_F = \mathcal{M}_{F-\ell}$ if ℓ is a loop. Moreover, $A_* = B_*$ and $d_*^A = -d_*^B$ (by the definition of the signed per-edge maps), so we have

$$\begin{array}{ccccccc} \longrightarrow & A_{i,j} & \xrightarrow{-d_{i,j}^B} & A_{i-1,j} & \xrightarrow{-d_{i-1,j}^B} & A_{i-2,j} & \longrightarrow \\ & & \searrow \text{id} & & \searrow \text{id} & & \\ \longrightarrow & B_{i+1,j} & \xrightarrow{d_{i+1,j}^B} & B_{i,j} & \xrightarrow{d_{i,j}^B} & B_{i-1,j} & \longrightarrow \end{array}$$

where

$$C_{i,j}(G) = A_{i-1,j} \oplus B_{i,j} \quad \text{and} \quad d_{i,j} = \begin{bmatrix} -d_{i-1,j}^B & 0 \\ \text{id} & d_{i,j}^B \end{bmatrix}.$$

In other words, $C_*(G)$ is the mapping cone of the negative identity map $-\text{id} : A_* \rightarrow B_*$, so $C_*(G)$ is exact by Exercise 1.5.1 in [Weibel 1995].

We supply the details below. Since $d_{i,j}(a, b) = (-d_{i-1,j}^B(a), a + d_{i,j}^B(b))$, then

$$\ker d_{i,j} = \{(a, b) \mid a = -d_{i,j}^B(b) \in \text{im } d_{i,j}^B \subseteq \ker d_{i-1,j}^B\} = \{(-d_{i,j}^B(b), b) \mid b \in B_{i,j}\}.$$

On the other hand, $\text{im } d_{i+1} = \{(-d_i^B(x), x + d_{i+1}^B(y)) \mid (x, y) \in A_i \oplus B_{i+1}\}$. Since $\text{im } d_{i+1}^B \subseteq \ker d_i^B$, then $(0, d_{i+1}^B(y)) = (-d_i^B(z), z)$ for some $z \in \ker d_i^B \subseteq B_i$. Thus

$$\text{im } d_{i+1} = \{(-d_i^B(x + z), x + z) \mid x + z \in B_i\} = \ker d_i.$$

Therefore, $C_*(G)$ is exact, and $H_*(G) = 0$. \square

Proposition 3.2. *Let G be a multigraph with edges e and e' each incident to the vertices x and y . Then $H_*(G) = H_*(G - e')$.*

Proof. Suppose G has m edges $\{e_1, \dots, e_m\}$ and $e_m = e$. Let $Z = G - e'$. Since Z is a subgraph of G , then $C_i(Z) \subseteq C_i(G)$ for all $i \geq 0$. Define $C_i(G, Z) = C_i(G)/C_i(Z)$. As the differential $d_i : C_i(G) \rightarrow C_{i-1}(G)$ sends $C_i(Z)$ to $C_{i-1}(Z)$, then there is an induced complex

$$\cdots \longrightarrow C_i(G, Z) \xrightarrow{d_i} C_{i-1}(G, Z) \longrightarrow \cdots$$

Define $H_i(G, Z) = \ker d_i / \text{im } d_{i+1}$. There are short exact sequences

$$0 \longrightarrow C_i(Z) \hookrightarrow C_i(G) \longrightarrow C_i(G, Z) \longrightarrow 0$$

for each $i \geq 0$, and these induce the long exact sequence of homology

$$\cdots \longrightarrow H_i(Z) \longrightarrow H_i(G) \longrightarrow H_i(G, Z) \longrightarrow H_{i-1}(Z) \longrightarrow \cdots$$

We shall show that $H_{i,j}(G, Z) = 0$ for all $i, j \geq 0$, from whence it follows that $H_*(G) = H_*(Z)$.

To compute relative homology $H_*(G, Z)$, we only need to consider the states in $Q(G)$ which contain e' . Let (A_*, d_*^A) denote the chain complex obtained by restricting $(C_*(G), d_*)$ to the modules indexed by the states which contain both e' and e , and let (B_*, d_*^B) denote the chain complex obtained by restricting $(C_*(G), d_*)$ to the modules indexed by the states which contain e' but not e . The only per-edge maps from modules in A_* to modules in B_* are of the form $\mathcal{M}_F \rightarrow \mathcal{M}_{F-e}$, and each of these maps is the identity map because $e' \in F$ implies $\mathcal{M}_F = \mathcal{M}_{F-e}$. Similar to the case of loops, we see from this that $C_*(G, Z)$ is the mapping cone of the negative identity map $-\text{id} : A_*[1] \rightarrow B_*$, so $C_*(G, Z)$ is exact, and hence $H_*(G, Z) = 0$. \square

3.2. Disjoint unions. Given graphs A and B , let $A + B$ denote their disjoint union. Since the power sum symmetric functions are multiplicative, one can deduce from this that $X_{A+B} = X_A X_B$. The lift to homology is the next result. The specialization of the Frobenius series at $q = t = 1$ recovers the multiplicative property of the chromatic symmetric function of a disjoint union of graphs.

Proposition 3.3. *For $i, j \geq 0$,*

$$H_{i,j}(A + B) = \bigoplus_{\substack{p+r=i \\ q+s=j}} \text{Ind}_{\mathfrak{S}_A \times \mathfrak{S}_B}^{\mathfrak{S}_{A \cup B}} (H_{p,q}(A) \otimes H_{r,s}(B)).$$

Proof. This is essentially given by the Künneth formula. Let $G = A + B$, and suppose we enumerate the edges of G by enumerating the edges of B first, then followed by the edges of A . Now, $Q(G) \cong Q(A) \times Q(B)$, so for each $i \geq 0$,

$$C_i(G) = \bigoplus_{\substack{F \in Q(G): \\ |F|=i}} \mathcal{M}_F = \bigoplus_{\substack{(F_A, F_B) \in Q(A) \times Q(B): \\ |F_A| + |F_B| = i}} \text{Ind}_{\mathfrak{S}_A \times \mathfrak{S}_B}^{\mathfrak{S}_{A \cup B}} (\mathcal{M}_{F_A} \otimes \mathcal{M}_{F_B}).$$

For the moment, we consider the $(\mathfrak{S}_A \times \mathfrak{S}_B)$ -modules

$$\tilde{C}_i(G) = \bigoplus_{\substack{(F_A, F_B) \in Q(A) \times Q(B): \\ |F_A| + |F_B| = i}} (\mathcal{M}_{F_A} \otimes \mathcal{M}_{F_B}),$$

so that $\tilde{C}_{i,j}(G) = \bigoplus_{\substack{p+r=i \\ q+s=j}} (C_{p,q}(A) \otimes C_{r,s}(B))$, and the chain maps are given by

$$\tilde{d}_{i,j}(c_A \otimes c_B) = d_{p,q}^A(c_A) \otimes c_B + (-1)^p c_A \otimes d_{r,s}^B(c_B).$$

These modules are free and hence are flat, so by [Weibel 1995, Theorem 3.6.3], there is an exact sequence

$$0 \rightarrow \bigoplus_{\substack{p+r=i \\ q+s=j}} (\tilde{H}_{p,q}(A) \otimes \tilde{H}_{r,s}(B)) \rightarrow \tilde{H}_{i,j}(G) \rightarrow \bigoplus_{\substack{p+r=i-1 \\ q+s=j}} \text{Tor}_1(\tilde{H}_{p,q}(A), \tilde{H}_{r,s}(B)) \rightarrow 0.$$

Moreover, there is zero torsion, again due to freeness. Therefore,

$$(12) \quad \tilde{H}_{i,j}(G) \cong \bigoplus_{\substack{p+r=i \\ q+s=j}} (\tilde{H}_{p,q}(A) \otimes \tilde{H}_{r,s}(B)).$$

Lastly, induce these to $\mathfrak{S}_{A \cup B}$ -modules to get the desired result. \square

Corollary 3.4. $\text{Frob}_{A+B}(q, t) = \text{Frob}_A(q, t) \cdot \text{Frob}_B(q, t)$. \square

Example 3.5. If G is a graph with n vertices, then the homology of the disjoint union of G with a single vertex is obtained by branching. If $H_{i,j}(G) = \bigoplus_{\lambda} (\mathcal{S}^{\lambda})^{\oplus m_{\lambda}}$, then

$$H_{i,j}(G + \bullet) = \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} H_{i,j}(G) \cong \bigoplus_{\mu} (\mathcal{S}^{\mu})^{\oplus m_{\lambda}},$$

where the sum is over all partitions μ which can be obtained by adding a box to the partitions λ indexing the irreducible factors of $H_{i,j}(G)$.

4. CATEGORIFYING THE RECURSIVE FORMULA FOR X_G

If the graph G has n vertices, then X_G is homogeneous of degree n , so it follows that X_G does not satisfy a deletion-contraction type recurrence. However, if G contains triangles, then the following result provides a method for expressing X_G as a linear combination of chromatic symmetric functions of graphs with fewer edges.

Theorem 4.1 ([Guay-Paquet 2013, Proposition 3.1] and [Orellana and Scott 2014, Theorem 3.1]). *Let G be a graph where the edges $e_1, e_2, e_3 \in E(G)$ form a triangle. Then*

$$X_G = X_{G-e_1} + X_{G-e_2} - X_{G-e_1-e_2}.$$

□

The homological analogue of this three term recurrence equation is a Mayer-Vietoris type long exact sequence. Let $A = G - e_1$, $B = G - e_2$, and let

$$(13) \quad C_i(A \cup B) = \bigoplus_{\substack{|F|=i, \\ F \in Q(A) \cup Q(B)}} \mathcal{M}_F$$

be the submodule of $C_i(G)$ that is the direct sum of the modules \mathcal{M}_F where F is a state of the subgraph A or a state of the subgraph B . The differential $d : C_i(G) \rightarrow C_{i-1}(G)$ sends $C_i(A \cup B)$ to $C_{i-1}(A \cup B)$, so $C_*(A \cup B)$ forms a chain complex. Let $H_*(A \cup B)$ be the bigraded homology resulting from $C_*(A \cup B)$.

Lemma 4.2. $H_*(G) \cong H_*(A \cup B)$.

Proof. We shall show that the inclusion $\iota : C_*(A \cup B) \hookrightarrow C_*(G)$ induces isomorphisms on homology. That is, we shall define a chain map $f_* : C_*(G) \rightarrow C_*(A \cup B)$ and prism operators $s_i : C_i(G) \rightarrow C_{i+1}(G)$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{i+1}(G) & \xrightarrow{d_{i+1}} & C_i(G) & \xrightarrow{d_i} & C_{i-1}(G) \longrightarrow \cdots \\ & & \downarrow \iota \circ f & \swarrow s_i & \downarrow \iota \circ f & \swarrow s_{i-1} & \downarrow \iota \circ f \\ \cdots & \longrightarrow & C_{i+1}(G) & \xrightarrow{d_{i+1}} & C_i(G) & \xrightarrow{d_i} & C_{i-1}(G) \longrightarrow \cdots \end{array}$$

such that $\text{id}_{C_*(G)} - \iota \circ f = ds + sd$ and $f \circ \iota = \text{id}_{C_*(A \cup B)}$, whence it follows that $H_*(G) \cong H_*(A \cup B)$.

To define f_* , partition the states of G in the following way. Suppose G has m edges $\{e_1, \dots, e_m\}$ and e_1, e_2, e_3 form a triangle T . Then G has 2^{m-3} states containing T . Each such state X indexes a Boolean sublattice of $Q(G)$ isomorphic to 2^T :

$$(14) \quad \begin{array}{ccccc} & & X_1 & \longrightarrow & X_{12} \\ & \nearrow & & \searrow & \nearrow \\ X & \longrightarrow & X_2 & & X_{13} \longrightarrow X_{123} \\ & \searrow & & \nearrow & \searrow \\ & & X_3 & \longrightarrow & X_{23} \end{array}$$

where $X_i = X - e_i$, $X_{ij} = X - e_i - e_j$, and $X_{123} = X - T$. We note that $Q(G) \cong 2^T \times 2^{m-3}$ as Boolean lattices, and this observation will be important for the computations to come.

As the differentials d_* are defined on the modules indexed by the states of G , we can do the same to define f_* and s_* . Partition the states of G into 2^{m-3} sublattices, each indexed by an $X \supseteq T$. On each such sublattice, define f_* on the eight corresponding modules as follows.

- (i) If $F = X$, let $f = 0$ on \mathcal{M}_F .
- (ii) The restriction of f to $\mathcal{M} = \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3}$ is given by

$$f : \mathcal{M} \rightarrow \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} : (x, y, z) \mapsto (x - z, y + z).$$

- (iii) If F is one of X_{12}, X_{13}, X_{23} or X_{123} , let f be the identity map on \mathcal{M}_F .

We remark that since T is a triangle, then deleting any one of the edges e_1, e_2 or e_3 will not disconnect X , and therefore, $\mathcal{M}_X = \mathcal{M}_{X_i}$ for $i = 1, 2, 3$.

We now check that f_* is a chain map. Suppose $X = \{e_1, e_2, e_3, e_{i_1}, \dots, e_{i_r}\}$, and consider the restriction of f to \mathcal{M}_X . Then

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3} \oplus \mathcal{N} \\ f_{r+3}=0 \downarrow & & \downarrow f_{r+2} \\ 0 & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \end{array}$$

where $\mathcal{N} = \bigoplus_{j=1}^r \mathcal{M}_{X_{i_j}}$. For $1 \leq j \leq r$, each $X_{i_j} \supseteq T$, so $f|_{X_{i_j}} = 0$. Depending on the parity of m , there are two different possibilities of signs. So for any $x \in \mathcal{M}_X$,

$$f_{r+2}(d_{r+3}(x)) = f_{r+2}(\pm x, \mp x, \pm x, n) = 0 = d_{r+3}(f_{r+3}(x)),$$

where $n \in \mathcal{N}$ is the image of x under d_{r+3} .

Next, consider the restriction of f to \mathcal{M}_{X_3} . Then

$$\begin{array}{ccc} \mathcal{M}_{X_3} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} \oplus \mathcal{N} \\ f_{r+2} \downarrow & & \downarrow f_{r+1} \\ \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} \oplus \mathcal{P} \end{array}$$

where $\mathcal{N} = \bigoplus_{j=1}^r \mathcal{M}_{X_{3,i_j}}$ is a direct sum of modules such that each summand is indexed by a state F' that contains e_1 and e_2 but not e_3 , and $\mathcal{P} = \bigoplus_{j=1}^r (\mathcal{M}_{X_{1,i_j}} \oplus \mathcal{M}_{X_{2,i_j}})$. Observe that $\mathcal{M}_{X_{3,i_j}} = \mathcal{M}_{X_{1,i_j}} = \mathcal{M}_{X_{2,i_j}}$ since they have the same connected graph components, so that when f is restricted to $\mathcal{M}_{X_{3,i_j}}$, then by definition,

$$f : \mathcal{M}_{X_{3,i_j}} \rightarrow \mathcal{M}_{X_{1,i_j}} \oplus \mathcal{M}_{X_{2,i_j}} : z \mapsto (-z, z).$$

Therefore, for any $x \in \mathcal{M}_{X_3}$,

$$\begin{aligned}
d_{r+2}(f_{r+2}(x)) &= d_{r+2}(-x, x) \\
&= \left((\mp(-x))^{\mu_1} + (\mp x)^{\mu_1}, (\pm(-x))^{\mu_2}, (\pm x)^{\mu_3}, \sum_{j=1}^r (-z_j, z_j) \right) \\
&= \left(0, (\mp x)^{\mu_2}, (\pm x)^{\mu_3}, \sum_{j=1}^r (-z_j, z_j) \right) \\
&= f_{r+1} \left((\mp x)^{\mu_2}, (\pm x)^{\mu_3}, \sum_{j=1}^r z_j \right) \\
&= f_{r+1}(d_{r+2}(x)),
\end{aligned}$$

where z_j denotes the image of x under d_{r+2} in \mathcal{M}_{3,i_j} .

The remaining cases to be checked are similar to each other. Consider the restriction of f to $\mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2}$. Then

$$\begin{array}{ccc}
\mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} \oplus \mathcal{N} \\
f_{r+2}=\text{id} \downarrow & & \downarrow f_{r+1}=\text{id} \\
\mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} \oplus \mathcal{N}
\end{array}$$

where \mathcal{N} is a direct sum of modules such that each summand $\mathcal{M}_{F'}$ is indexed by a state F' that contains e_3 and one of e_1 or e_2 . So $f|_{\mathcal{N}} = \text{id}$, and thus $f_{r+1}d_{r+2} = d_{r+2}f_{r+2}$.

Next, consider the restriction of f to $\mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}}$. Then

$$\begin{array}{ccc}
\mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}} \oplus \mathcal{N} \\
f_{r+1}=\text{id} \downarrow & & \downarrow f_r=\text{id} \\
\mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}} \oplus \mathcal{N}
\end{array}$$

where \mathcal{N} is a direct sum of modules such that each summand $\mathcal{M}_{F'}$ is indexed by a state F' that contains at most one of the edges e_1, e_2 or e_3 . So $f|_{\mathcal{N}} = \text{id}$, and thus $f_r d_{r+1} = d_{r+1} f_{r+1}$.

Lastly, consider the restriction of f to $\mathcal{M}_{X_{123}}$. Then

$$\begin{array}{ccc}
\mathcal{M}_{X_{123}} & \xrightarrow{d_r} & \mathcal{N} \\
f_r=\text{id} \downarrow & & \downarrow f_{r-1}=\text{id} \\
\mathcal{M}_{X_{123}} & \xrightarrow{d_r} & \mathcal{N}
\end{array}$$

where \mathcal{N} is a direct sum of modules such that each summand $\mathcal{M}_{F'}$ is indexed by a state F' that does not contain any of the edges e_1, e_2 or e_3 . So $f|_{\mathcal{N}} = \text{id}$, and thus $f_{r-1}d_r = d_r f_r$.

Therefore, f_* is a chain map.

It is straightforward to show that $f \circ \iota = \text{id}_{C_*(A \cup B)}$. It suffices to check this when restricted to each of the 2^{m-3} sublattices indexed by a state $X \supseteq T$. Note that $f_{r+2}(\iota(x, y)) = f_{r+2}(x, y, 0) = (x, y)$.

$$\begin{array}{ccccccc}
 0 & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}} \\
 \downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
 \mathcal{M}_X & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}} \\
 \downarrow f_{r+3}=0 & & \downarrow f_{r+2} & & \downarrow f_{r+1}=\text{id} & & \downarrow f_r=\text{id} \\
 0 & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}}
 \end{array}$$

Finally, we show that $\iota \circ f$ is chain homotopic to the identity. Again, partition the states of G into 2^{m-3} sublattices, each indexed by an $X \supseteq T$, and define s_* on the eight corresponding modules.

Suppose $X = \{e_1, e_2, e_3, e_{i_1}, \dots, e_{i_r}\}$, and without loss of generality, we may assume that X is connected, for otherwise, define s on the component of X containing T , then extend s to be identity on the remaining components, and finally induce $s \otimes \text{id}$ to a map on \mathfrak{S}_n -modules.

- (i) Since T is a triangle, then deleting the edge e_3 will not disconnect X , and therefore, $\mathcal{M}_X = \mathcal{M}_{X_3}$. If $F = X_3$, let $s_{r+2}(x) = \text{sgn}(d_\varepsilon)x$ for any $x \in X_3$, where $\text{sgn}(d_\varepsilon)$ is the sign of the per-edge map $d_\varepsilon : X \rightarrow X_3$ in $Q(G)$.
- (ii) If F is any of the seven remaining states, let $s|_F = 0$.

The following diagram illustrates s_* when restricted to a sublattice indexed by some $X \supseteq T$. To compute $ds + sd$, one needs to direct sum all 2^{m-3} such sublattices.

$$\begin{array}{ccccccc}
 \mathcal{M}_X & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}} \\
 \downarrow f_{r+3}=0 & & \swarrow s_{r+2} & \downarrow f_{r+2} & \swarrow s_{r+1}=0 & \downarrow f_{r+1}=\text{id} & \swarrow s_r=0 \\
 0 & & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & & \mathcal{M}_{X_{123}} \\
 \downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
 \mathcal{M}_X & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3} & \xrightarrow{d_{r+2}} & \mathcal{M}_{X_{12}} \oplus \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} & \xrightarrow{d_{r+1}} & \mathcal{M}_{X_{123}}
 \end{array}$$

Since the prism operators are almost all zero, it is only necessary to check that $\text{id} - \iota \circ f = ds + sd$ when restricted to \mathcal{M}_X and \mathcal{M}_{X_3} ; the other cases are clear.

Consider the case $F = X$.

$$\begin{array}{ccc}
 & \mathcal{M}_X & \xrightarrow{d_{r+3}} \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3} \oplus \mathcal{N} \\
 & \downarrow f_{r+3}=0 & \\
 \mathcal{O} & \xleftarrow{s_{r+3}=0} & 0 \\
 & \downarrow \iota & \\
 \mathcal{O} & \xrightarrow{d_{r+4}} & \mathcal{M}_X
 \end{array}$$

where $\mathcal{N} = \bigoplus_{j=1}^r \mathcal{M}_{X_{i_j}}$ such that $X_{i_j} \supseteq T$, so $s|_{\mathcal{N}} = 0$ (and by definition s_{r+3} is nonzero only on the summand \mathcal{M}_{X_3}).

Thus, for all $x \in \mathcal{M}_X$, $(\text{id} - \iota \circ f)(x) = x$, and

$$(d_{r+4}s_{r+3} + s_{r+2}d_{r+3})(x) = 0 + s_{r+2}(\pm x, \mp x, \pm x, n) = \pm(\pm x) = x.$$

As for the case $F = X_3$,

$$\begin{array}{ccc}
 & \mathcal{M}_{X_3} & \xrightarrow{d_{r+2}} \mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}} \oplus \mathcal{N} \\
 & \downarrow f_{r+2} & \\
 & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} & \\
 & \downarrow \iota & \\
 \mathcal{M}_X & \xrightarrow{d_{r+3}} & \mathcal{M}_{X_1} \oplus \mathcal{M}_{X_2} \oplus \mathcal{M}_{X_3} \oplus \mathcal{P}
 \end{array}$$

where $\mathcal{N} = \bigoplus_{j=1}^r \mathcal{M}_{X_{3,i_j}}$ is a direct sum of modules such that each summand is indexed by a state F' that contains e_1 and e_2 but not e_3 , and $\mathcal{P} = \bigoplus_{j=1}^r \mathcal{M}_{X_{i_j}}$.

Note, in particular, that

$$\begin{array}{ccccc}
 \mathcal{M}_{X_3} & \xrightarrow{s_{r+2}=\text{sgn}(d_\varepsilon)\text{id}} & \mathcal{M}_X & \xrightarrow{\text{sgn}(d_\varepsilon)\text{id}} & \mathcal{M}_{X_3} \\
 x & \longmapsto & \text{sgn}(d_\varepsilon)x & \longmapsto & \text{sgn}(d_\varepsilon)x = x,
 \end{array}$$

so it follows that

$$d_{r+3}(s_{r+2}(x)) = d_{r+3}(\text{sgn}(d_\varepsilon)x) = \left(x, -x, x, \sum_{j=1}^r z_j \right),$$

where ε is the edge from X to X_3 in $Q(G)$, and z_j is image of x in $\mathcal{M}_{X_{3,i_j}}$ under d .

On the other hand, note that $\mathcal{M}_{X_{3,i_j}} = \mathcal{M}_{X_{i_j}}$ because they have the same connected graph components, so the per-edge maps $\mathcal{M}_{X_3} \rightarrow \mathcal{N}$ and $\mathcal{M}_X \rightarrow \mathcal{P}$ are identical. As well, it follows from this that s is the identity map when restricted to \mathcal{N} . Moreover, if η_j is the edge from X_{i_j} to X_{3,i_j} in $Q(G)$, then $\text{sgn}(d_{\eta_j}) = -\text{sgn}(d_\varepsilon)$, since $i_j > 3$ by

assumption. Also, we mention that by definition, s is zero on $\mathcal{M}_{X_{13}} \oplus \mathcal{M}_{X_{23}}$, so

$$s_{r+1}(d_{r+2}(x)) = s_{r+1}\left((-x)^{\mu_2}, x^{\mu_3}, \sum_{j=1}^r z_j\right) = \left(0, 0, 0, -\sum_{j=1}^r z_j\right),$$

and therefore,

$$(d_{r+3}s_{r+2} + s_{r+1}d_{r+2})(x) = (x, -x, x, 0).$$

Lastly, for all $x \in \mathcal{M}_{X_3}$, $(\text{id} - \iota \circ f)(x) = (0, 0, x, 0) - (-x, x, 0) = (x, -x, x, 0)$. So $\text{id} - \iota \circ f = ds + sd$ when restricted to \mathcal{M}_{X_3} .

This completes the proof that $\text{id}_{C_*(G)}$ is homotopic to $\iota \circ f$. Therefore, $H_*(G) \cong H_*(A \cup B)$. \square

Lemma 4.3. *Let G be a graph where the edges $e_1, e_2, e_3 \in E(G)$ form a triangle. Let $A = G - e_1$ and $B = G - e_2$. For $i \geq 0$, there are short exact sequences*

$$0 \longrightarrow C_i(A \cap B) \xrightarrow{\varphi} C_i(A) \oplus C_i(B) \xrightarrow{\psi} C_i(A \cup B) \longrightarrow 0$$

where $\varphi(x) = (x, -x)$, and $\psi(a, b) = a + b$.

Proof. We have $\ker \varphi = 0$, since $\varphi(x) = (x, -x) = (0, 0)$ implies $x = 0$. Next, since $(\psi \circ \varphi)(x) = \psi(x, -x) = x - x = 0$, then $\text{im } \varphi \subseteq \ker \psi$. Also, $\ker \psi \subseteq \text{im } \varphi$, since if $\psi(a, b) = a + b = 0$, then $a = -b \in C_i(B)$, so $a \in C_i(A \cap B)$, and $(a, b) = \varphi(a) \in \text{im } \varphi$. Lastly, $\text{im } \psi = C_i(A \cup B)$ since ψ acts as identity when restricted to either $C_i(A)$ or $C_i(B)$. \square

Theorem 4.4. *Let G be a graph where the edges $e_1, e_2, e_3 \in E(G)$ form a triangle. Let $A = G - e_1$ and $B = G - e_2$. Then there is a long exact sequence*

$$\cdots \longrightarrow H_i(A \cap B) \xrightarrow{\varphi} H_i(A) \oplus H_i(B) \xrightarrow{\psi} H_i(G) \xrightarrow{\delta} \cdots \longrightarrow H_0(G) \longrightarrow 0.$$

Proof. The short exact sequences of chain complexes in Lemma 4.3 induce a long exact sequence in homology. By Lemma 4.2, since $H_*(A \cup B) \cong H_*(G)$, then the result follows. \square

Corollary 4.5.

$$\text{Frob}_G(q, 1) = \text{Frob}_{G-e_1}(q, 1) + \text{Frob}_{G-e_2}(q, 1) - \text{Frob}_{G-e_1-e_2}(q, 1).$$

Proof. The maps in Theorem 4.4 are degree-preserving, so applying the Frobenius characteristic map gives the result. \square

Remark 4.6. Further setting $q = 1$ in the above Corollary recovers the result in Theorem 4.1.

Example 4.7. Consider $G = K_3$. Theorem 4.1 gives

$$\begin{array}{c} \text{Diagram of } K_3 \text{ with edges } e_1, e_2, e_3 \end{array} = \begin{array}{c} \text{Diagram of } A \end{array} + \begin{array}{c} \text{Diagram of } B \end{array} - \begin{array}{c} \text{Diagram of } A \cap B \end{array}$$

$$6s_{111} = (s_{21} + 4s_{111}) + (s_{21} + 4s_{111}) - (2s_{21} + 2s_{111})$$

To condense notation, let $H_*(Z) = H_*(A \cap B)$ and $H_*(Y) = H_*(A) \oplus H_*(B)$. Using the computations from Sections 2.14, 5.1 and Section 5.2 combined with Proposition 3.3, the long exact sequence for $G = K_3$ is

$$H_2Z \rightarrow H_2Y \rightarrow H_2G \rightarrow H_1Z \longrightarrow H_1Y \longrightarrow H_1G \longrightarrow H_0Z \rightarrow H_0Y \rightarrow H_0G \rightarrow 0$$

$$\begin{array}{cccccccccccccccc}
 0 & \longrightarrow & 2\mathcal{S}^{111} & \longrightarrow & 2\mathcal{S}^{111} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{S}^{21} + \mathcal{S}^{111} & \longrightarrow & 2\mathcal{S}^{21} + 4\mathcal{S}^{111} & \longrightarrow & \mathcal{S}^{21} + 3\mathcal{S}^{111} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{S}^{21} & \longrightarrow & \mathcal{S}^{21} + \mathcal{S}^{111} & \longrightarrow & 2\mathcal{S}^{111} & \longrightarrow & \mathcal{S}^{111} & \longrightarrow & 0
 \end{array}$$

In terms of the Frobenius series $\text{Frob}_G(q, 1)$,

$$\begin{array}{rclclcl}
 \text{Frob}_{K_3}(q, 1) & = & \text{Frob}_A(q, 1) & + & \text{Frob}_B(q, 1) & - & \text{Frob}_{A \cap B}(q, 1) \\
 2q^2s_{111} & = & q^2s_{111} & + & q^2s_{111} & - & 0 \\
 q(s_{21} + 3s_{111}) & = & q(s_{21} + 2s_{111}) & + & q(s_{21} + 2s_{111}) & - & q(s_{21} + s_{111}) \\
 s_{111} - s_{21} & = & s_{111} & + & s_{111} & - & (s_{21} + s_{111})
 \end{array}$$

5. EXAMPLES

We include computations of chromatic symmetric homology for several small graphs.

5.1. Edge-less graphs. Let G be the edge-less graph on n vertices. In this case, G only has one state, and the \mathfrak{S}_n -module associated to the state is

$$\text{Ind}_{\mathfrak{S}_1 \times \dots \times \mathfrak{S}_1}^{\mathfrak{S}_n} (\mathcal{S}^1 \otimes \dots \otimes \mathcal{S}^1) = \bigoplus_{\lambda \vdash n} f^\lambda \mathcal{S}^\lambda,$$

where $f^\lambda = \chi^\lambda(1^n) = \dim(\mathcal{S}^\lambda)$ is the number of standard Young tableaux of shape λ . Thus $H_{00} = \bigoplus_{\lambda \vdash n} f^\lambda \mathcal{S}^\lambda$, and the bigraded Frobenius series $\text{Frob}_G(q, t) = \sum_{\lambda \vdash n} f^\lambda s_\lambda = p_1^n = X_G$ is independent of q and t .

5.2. A single edge. Let $G = K_2$. The diagram of states together with the signed per-edge map is

$$\begin{array}{ccc}
 \bullet \text{---} \bullet & \xrightarrow{d_*} & \bullet \quad \bullet \\
 1 & & 0
 \end{array}$$

The graded \mathfrak{S}_2 -modules corresponding to the states are:

$$\begin{array}{ccc} & \mathcal{S}^{11} & \\ \uparrow & & \\ \mathcal{S}^2 & \xrightarrow{d_{10}} & \text{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_1}^{\mathfrak{S}_2} (\mathcal{S}^1 \otimes \mathcal{S}^1) \end{array}$$

Recall that $\mathfrak{S}_2 = \langle s_1 \mid s_1^2 = e \rangle$. Then $\text{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_1}^{\mathfrak{S}_2} (\mathcal{S}^1 \otimes \mathcal{S}^1) \cong \mathbb{C}[\mathfrak{S}_2] \cong \mathcal{S}^2 \oplus \mathcal{S}^{11} = \text{span}_{\mathbb{C}}\{e, s_1\}$. Choosing $\mathcal{S}^2 = \text{span}_{\mathbb{C}}\{e + s_1\}$ and $\mathcal{S}^{11} = \text{span}_{\mathbb{C}}\{e - s_1\}$, then

$$d_{10} : \text{span}_{\mathbb{C}}\{e + s_1\} \rightarrow \text{span}_{\mathbb{C}}\{e, s_1\}$$

is simply the inclusion map. Thus $\ker d_{11} \cong \mathcal{S}^{11}$, $\text{im } d_{10} \cong \mathcal{S}^2$, and $\ker d_{00} \cong \mathcal{S}^2 \oplus \mathcal{S}^{11}$. In summary, the chromatic symmetric homology of graded \mathfrak{S}_2 -modules for $G = K_2$ is

$$\begin{array}{c|cc} 1 & \mathcal{S}^{11} & \\ 0 & 0 & \mathcal{S}^{11} \\ \hline & H_1(K_2) & H_0(K_2) \end{array}$$

and the bigraded Frobenius series is $\text{Frob}_{K_2}(q, t) = qts_{11} + s_{11} = (1 + qt)s_{11}$.

5.3. Some small trees. While the chromatic polynomial χ_G does not distinguish trees, it is conjectured that the chromatic symmetric function is a complete invariant for trees. In this example, we state results in homology for trees up to four vertices.

Example 5.1. Consider the two trees on four vertices. Let

$$P_3 = \bullet \text{---} \bullet \text{---} \bullet \quad P_4 = \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad T_4 = \begin{array}{c} \bullet \\ \diagdown \\ \bullet \text{---} \bullet \\ \diagup \\ \bullet \end{array}$$

The chromatic symmetric homology of graded \mathfrak{S}_n -modules for each of these graphs is

$$\begin{array}{c|ccc} 2 & \mathcal{S}^{111} & & \\ 1 & 0 & \mathcal{S}^{21} \oplus (\mathcal{S}^{111})^{\oplus 2} & \\ 0 & 0 & 0 & \mathcal{S}^{111} \\ \hline & H_2(P_3) & H_1(P_3) & H_0(P_3) \end{array}$$

$$\begin{array}{c|cccc} 3 & \mathcal{S}^{1111} & & & \\ 2 & 0 & \mathcal{S}^{22} \oplus (\mathcal{S}^{211})^{\oplus 2} \oplus (\mathcal{S}^{1111})^{\oplus 3} & & \\ 1 & 0 & 0 & \mathcal{S}^{22} \oplus (\mathcal{S}^{211})^{\oplus 2} \oplus (\mathcal{S}^{1111})^{\oplus 3} & \\ 0 & 0 & 0 & 0 & \mathcal{S}^{1111} \\ \hline & H_3(P_4) & H_2(P_4) & H_1(P_4) & H_0(P_4) \end{array}$$

and

3	\mathcal{S}^{1111}			
2	0	$(\mathcal{S}^{211})^{\oplus 2} \oplus (\mathcal{S}^{1111})^{\oplus 3}$		
1	0	\mathcal{S}^{22}	$\mathcal{S}^{31} \oplus \mathcal{S}^{22} \oplus (\mathcal{S}^{211})^{\oplus 3} \oplus (\mathcal{S}^{1111})^{\oplus 3}$	
0	0	0	\mathcal{S}^{22}	\mathcal{S}^{1111}
	$H_3(T_4)$	$H_2(T_4)$	$H_1(T_4)$	$H_0(T_4)$

Their Frobenius series are

$$\text{Frob}_{P_3}(q, t) = qts_{21} + (1 + qt)^2 s_{111},$$

$$\text{Frob}_{P_4}(q, t) = qt(1 + qt)s_{22} + 2qt(1 + qt)s_{211} + (1 + qt)^3 s_{1111},$$

$$\text{Frob}_{T_4}(q, t) = qts_{31} - t(1 - q + qt)s_{22} + qt(3 + 2qt)s_{211} + (1 + qt)^3 s_{1111}.$$

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(Radmila Sazdanović) NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695
E-mail address: rsazdanovic@math.ncsu.edu

(Martha Yip) UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506
E-mail address: martha.yip@uky.edu